Mathematics Education Research Journal 2006, Vol. 18, No. 1, 103–123

The Case of Bruce: A Teacher's Model of his Students' Algebraic Thinking About Equivalent Expressions

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The purpose of this article is to describe a middle school mathematics teacher's model of his students' responses to algebraic tasks involving equivalent expressions and the distributive property. The teacher engaged in two model-eliciting activities designed for teachers by creating a library of his students' work and an accompanying "Ways of Thinking"[WOT] sheet (Doerr & Lesh, 2003). These activities were designed to help reveal the teachers' models of students' algebraic thinking and to promote the development of that model. Results of the analysis showed that the teacher developed a clearer understanding of the role of a variable in algebraic instruction. The teacher employed visual strategies for the first time and began to perceive their usefulness in helping students understand the equivalence of two expressions.

Many researchers have called for instruction that focuses on helping students make connections between procedural and structural approaches to algebra; yet research on the teaching of algebra shows that instruction tends to emphasise procedural approaches rather than structural ones (Attorps, 2005; Kieran, 1992; Menzel & Clark, 1998, 1999; Nathan & Koedinger, 2000). Students' difficulties in making the cognitive leap from arithmetic to algebra are, at times, related to instructional strategies (Kieran, 1992). However, as Kieran (1992) noted, "there is a scarcity of research emphasizing the role of the classroom teacher in algebra instruction" (p. 395). Although the number of research studies on the teaching of algebra and algebra teachers has grown over the last decade (Kieran, 2006), the research base on teachers' knowledge regarding algebra is still quite limited (Attorps, 2005; Doerr, 2004; Menzel & Clark, 1999; Rand Mathematics Study Panel, 2003).

The calls for reform in algebraic instruction (Kaput, 2000; MacGregor, 2004; National Council of Teachers of Mathematics [NCTM], 2000; Warren, 2003) advocate a shift from a focus on procedural aspects such as simplifying expressions and solving equations to developing conceptual knowledge through a variety of expanded views of algebra. These may include approaches to algebra that focus on functions, generalising patterns, learning through problem solving, or a modelling approach (Bednarz, 2001, Bednarz, Kieran, & Lee, 1996). These changes in algebraic instruction often include the use of contextual problems. Research has shown that implementing reform-based instructional programs can successfully advance students' conceptual thinking and skills, but such programs are not always implemented as intended (Hiebert, 1999; Schifter, 1996; Shepard, 2000). Unarguably, there exists a need for more effective models of how teachers interpret the learning process (Ball, 1997; Menzel & Clark, 1998, 1999; Shepard, 2000). Currently, US reform efforts also call for teachers “to analyze what they and their students are doing and consider how those actions are affecting students’ learning” (NCTM, 2000, p. 19). One approach is to have teachers examine students’ written work. The use of student work samples (Chamberlin, 2005; Doerr, in press; NCTM, 2000), including students’ responses to open-ended questions (Moskal & Magone, 2000; Van den Heuvel-Panhuizen, 1994), offers teachers the possibility of detailed information from which to examine students’ reasoning processes. These studies collectively reveal that when teachers examine student work, not all teachers acquire the same information, nor do teachers necessarily interpret students’ work in consistent ways.

The primary goal of the study discussed in this article was to focus on the nature of teachers’ models or systems of interpretations about their students’ algebraic thinking as the students engaged with a series of tasks on equivalent expressions and the distributive property. The study was set within the context of a reform-based curriculum that promoted the use of spatial representations to help students understand equivalent algebraic expressions. The aim of this study was to answer the following question: when students solve tasks on equivalent expressions, what information do teachers acquire about their students’ algebraic thinking, and how do they interpret that information?

Theoretical framework

A models and modelling perspective of teacher development guided the development of this study and framed the examination of the way teachers think in the context of their work. What teachers do is inherently complex. To make sense of complex situations, teachers need to develop systems of interpretation, or models, that account for their experiences (Lesh, Doerr, Carmona, & Hjalmarson, 2003). A modelling perspective of teacher development focuses upon the ways teachers think about and interpret their practice. This perspective is based upon the premise that:

> …it is not enough to see what a teacher does, we need to understand how and why the teacher was thinking in a given situation, that is, interpreting the salient features of the event, integrating them with past experiences, and anticipating actions, consequences, and subsequent interpretations. (Lesh, Doerr, Carmona, & Hjalmarson, 2003, p. 127).

This perspective incorporates the use of model-eliciting (or thought-revealing) activities for teachers to help account for the evolving nature of their own learning, and attempts to help them become reflective of their teaching efforts, as well as begin to recognise the multiple ways their students interpret mathematical problems.

Prior research posits that model development is a non-linear, cyclic process (Doerr & Lesh, 2003). In this study, teachers were engaged in their own model development while students were engaged in a series of open-ended tasks involving equivalent expressions and the distributive property (described in detail later in the article). Since little is known about teachers' knowledge concerning algebraic instruction, the goal was to describe the development of teachers' models about their students' algebraic thinking. In essence, the model-eliciting activities for teachers were designed to unmask the significant mathematical concepts behind the series of tasks. The models and modelling framework drew upon the mathematical knowledge teachers possessed and used that as the base to engage teachers in expressing, revising, and refining their knowledge. The teachers' models then served as interpretive, and explanatory frameworks to help make sense of their students' mathematical thinking.

Prior research informing the study

Research on connectedness between lessons involving equivalent expressions has shown that neither novice nor expert teachers used spatial arrangements to help students see that two expressions might be equivalent (Even, Tirosh, & Robinson, 1993). Tirosh, Even, and Robinson (1998) studied teachers' knowledge involving equivalent expressions and the distributive property and found that novice teachers were unaware of students' tendencies to conjoin or finish expressions (e.g., simplifying 4x + 7 to equal 11x). Teachers often move to a conceptual or structural level of algebraic instruction before students are ready (Kieran, 1992). Even (1993) found that teachers focused on giving students rules without regard to their conceptual understanding. Other studies have shown that many algebra teachers report that they value conceptual understanding, but in practice they emphasise specific skills and knowledge and, furthermore, algebra teachers may follow curricular instructional sequences without attending to the conceptual aspects of teaching algebra (Attorps, 2005; Menzel & Clark, 1998, 1999). Menzel and Clark (1999) concluded that unless teachers develop in-depth pedagogical content knowledge, they risk giving students a "limited, utilitarian concept of algebra" (p. 371). Attorps (2005) also found that textbooks may not provide sufficient information to promote teaching for conceptual understanding. Menzel (2001) further substantiated that algebra teachers often teach skills (such as graphing skills) with a focus on procedures instead of using the rich language of algebra which might afford teachers the opportunity to connect various representations central to students' conceptual algebraic understanding. Taken collectively, these studies indicate that beginning algebraic instruction does not tend to promote conceptual understanding. Despite the many difficulties cited in the research literature, isolated cases of dynamic algebraic instruction, where the procedural and conceptual nature of teaching algebra are linked, do exist (e.g., Chazan, 1999; Lloyd & Wilson, 1998).

Within the rather large body of research on middle school students' knowledge of algebra, the studies of students' knowledge of literal terms, variables, and algebraic expressions are most directly related to this study. Students find it complex and difficult to understand the structural features of algebra. In several studies it was noted that the transition from arithmetic to algebra is difficult (Kieran, 1992; MacGregor & Stacey, 1998; Stacey & MacGregor, 2000). Küchemann (1981) found that students preferred a single term solution. Kieran (1981) also found that students were uncomfortable with expressions standing alone. In addition, Booth (1988) and Stacey and MacGregor (2000) found that beginning algebra students wanted to find a numerical answer, not an algebraic one. Students' weak understanding of a variable allowed them to use letters to stand for physical objects rather than mathematical objects (Booth, 1988; Kaput, 1987; Küchemann, 1981; MacGregor & Stacey, 1993). Pimm (1987) illuminated this confusion that students face in understanding what letters represent. In some cases students experienced numbers as adjectives and letters as objects, as in the example $3y$ for three yachts. However, in making the transition to algebraic symbolism, students needed to see the letter "*a*" in $5a$ as standing for a number. Pimm noted that when teachers explained adding like terms by describing $5a + 2b$ as five apples plus two bananas (a "fruit salad approach"), misunderstandings about the role of a variable were perpetuated. MacGregor and Stacey (1997) also found that misleading curriculum materials detrimentally influenced students' understanding of a variable. In contrast, research involving reform-based programs has shown that middle school students were able to reference alternative representations (Brenner et al., 1997; Kaput, 2000; Langrell & Lannin, 2000).

Description of the study

Setting and participants

A cohort of teachers implementing a reformed middle school curriculum for the first time comprised the study's participants. The five middle school teachers were from two urban middle school settings situated in the northeastern United States. Four of the teachers had between 6 and 18 years of experience; the fifth was in her first year of teaching. Not all teachers were equally enthusiastic participants, nor did they all demonstrate the same amount of growth. All, however, developed increasingly powerful models of their students' algebraic thinking.

In this article, findings from one of the teachers, Bruce, are reported in order to describe the potential that can arise from the simultaneous implementation of a reformed curriculum and teacher participation in model-eliciting activities. Bruce had taught seventh and eighth grade mathematics for eighteen years in the same school setting.

Bruce was the one participant who taught the complete series of tasks examined in this study and the entire unit from *Say It With Symbols* (Lappan et al., 1998). He spent nine 50-minute periods teaching the series of tasks, which was about twice the amount of class time the other teachers spent. One teacher at Bruce's site was less committed to the new curriculum with its collaborative group structure and therefore spent only four days teaching the tasks. State testing interfered with the other three teachers' participation in the study. At Bruce's site, one teacher spent three days on the tasks. At the second site, both teachers spent four to five days on the tasks. The other four participants did not have the opportunity to revise and refine their models over time, as did Bruce. Consequently, Bruce is the focus of this paper.

Multi-tiered teaching experiment

To extend the teachers' knowledge into increasingly powerful models of classroom teaching, the teachers participated in a multi-tiered teaching experiment (Lesh & Kelly, 1999). To elicit teachers' models of their students' algebraic thinking, model-eliciting activities were used initially to perturb the teachers' thinking and promote knowledge development. In the multi-tiered research design, students, teachers, and researchers, played unique roles as shown in Figure 1.

Tier 3: Researcher Level	The researcher developed models to make sense of the teacher's model-eliciting activities. The researcher revealed her interpretations as she described and explained the teacher's models.
Tier 2: Teacher Level	The teacher, Bruce, developed shared tools in the form of "Ways of Thinking" sheets and libraries of student work. As he described, explained, and predicted students' behaviours via the shared tools, he constructed and refined his models to make sense of his students' algebraic thinking about equivalent expressions.
Tier 1: Student Level	Students worked on a series of algebraic tasks related to equivalent expressions from *Say It With Symbols* (Lappan et al., 1998), in which the goals included constructing and refining models (descriptions, explanations, justifications) that revealed how they interpreted the mathematical situation.

Figure 1. The design of multi-tiered teaching experiments (adapted from Lesh & Kelly, 1999, p. 198).

Taking into account the characteristics of a modelling perspective of teacher development presented earlier, results are presented about the model of one of the teachers from Tier 2, and focuses upon the ways Bruce thought about and interpreted his practice.

Data sources and analyses

The primary data sources used in this study consisted of two model-eliciting activities for teachers: "Ways of Thinking" (WOT) sheets, and a library of student work. Schorr & Lesh (2003) argued that a thought-revealing activity should result in a shared and reusable artifact that leaves behind a trail of documentation.

The teachers' model-eliciting activities

The first activity consisted of asking teachers to create WOT sheets (Doerr & Lesh, 2003). Preparing WOT sheets has been described as a task that engages teachers in anticipating and evaluating their students' mathematical ideas (Doerr & Lesh, 2003). In this study, teachers were asked to work on the WOT sheets on three occasions: before students undertook the series of algebraic tasks (described later in the text), immediately after implementing the tasks, and about a month later. Pre-service teachers were selected as an audience for the WOT sheets so that participants would assume the audience did not have extensive experience teaching the material. For each iteration of the WOT sheets, the teachers were directed to think of what might be useful to a pre-service teacher and write down such things as:

- hints about the students' mathematical thinking; and
- mistakes students might make (or made) in their mathematical thinking

The teachers were asked to work on the WOT sheets over time so as to have the opportunity to test, revise, and refine their thinking about their students' work.

The second activity consisted of asking the teachers to select, analyse, and interpret "exemplary and illuminating" (Doerr & Lesh, 2003, p. 136) samples of student work. The purpose of gathering student work was multifaceted:

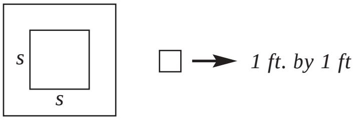
> As the teachers select, organise and compare student work, they reveal how they are seeing the students' mathematical ideas. This may lead to mismatches between their expectations of some students…It may lead to seeing students give mathematical interpretations of problem situations that the teacher had not seen. It is the resolution of such mismatches that provided the impetus for the development of teachers' knowledge. (Doerr & Lesh, 2003, p. 137).

The teachers were directed to save examples of student work that might be helpful to show a pre-service teacher how students in their own classrooms actually solved these problems. This library of student work samples also served as the focus of individual teacher interviews that were conducted after students had completed the algebraic tasks that are described next.

Students' algebraic tasks

Students (in each teacher's class) were asked to solve a series of tasks involving equivalent expressions. The tasks were drawn from the Connected Mathematics Project (CMP) book *Say It With Symbols* (Lappan et al., 1998). The students were first asked to find the number of 1-foot square tiles surrounding different sized square pools and then to find an equation for the number of tiles, N, that were needed to form a border for a square pool with sides of length s feet. Later, the students drew representations for the following problem:

Given a square pool as shown, draw a picture to illustrate the border of a square pool in four different ways:



a. 4(s + 1)
b. s + s + s + s + 4
c. 2s + 2(s + 2)
d. 4(s + 2) - 4
e. Explain why each expression in parts a-d is equivalent to 4s + 4. (adapted from Lappan et al., 1998, p. 22)

Finally, students applied the distributive property in the context of a rectangular pool divided into two sections.

Data gathering

Data for this study were gathered in three phases:

- Phase One : before students were asked to complete the algebraic tasks
- Phase Two: during and immediately after the students attempted the tasks; and
- Phase Three: the final exit interviews with teachers about a month after the students had completed the tasks.

Team meetings

The study began with team meetings at each site. Team discussions were held at each site during each of the three data gathering phases. During phase one, at the first team meeting, explanations were provided for: the study procedures, how to create WOT sheet, and how to develop a library of student work. Teachers' questions were also answered.

In subsequent team meetings, interview questions used for the individual interviews were discussed and the libraries of student work were examined. Team meetings were difficult to schedule given the constraints of the school day. At both sites, the only common planning time during the school day was during the lunch period.

The intent of the meetings was to promote discussion about students' ways of thinking about mathematical problems. None of the teachers had ever participated in team meetings of this nature before. They were able to see the types of questions other teachers selected to give as homework, class work, and for quizzes (short examinations) as well as the results of instruction.

In the final team meeting, each team was presented with additional algebraic problems related to the series of tasks students had been asked to solve. Three methods to explain adding like terms were offered to each team, and the methods that might be preferred by each teacher were discussed.

The first method (shown in Figure 2) involved a linear and area representation in which two figures represented the area of a rectangle, and two figures represented the length of a line. The area of the rectangle was in square centimetres and the length of the line was in centimetres.

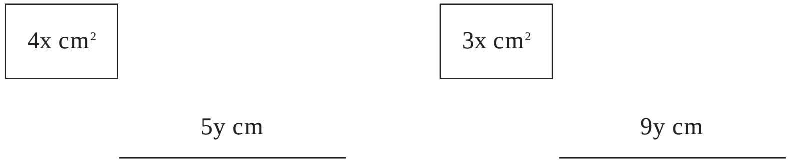


Figure 2. Area and linear units problem

The students were asked how to represent the total units.

For the second method a bank account problem was presented in which two depositors who had an established pattern of saving each week decided to open a joint account: “Ryan started a bank account with $50, and added $40 a week. Allison started a bank account with $90 and added $30 a week. If Ryan and Allison decided to start a joint account instead, how would you represent the total from both children?”

Finally, an “apples and oranges” method of explanation was presented.

The intent of this activity was to promote additional discussion on how to explain adding like terms. After this team discussion, teachers would have one more opportunity to add to their WOT sheets at the final exit individual interview.

Individual interviews and model-eliciting tasks

Each teacher was interviewed during each phase of the study. It was during the individual interviews that teachers created their WOT sheets and libraries of student work. In the first interview, each teacher created their first preliminary WOT sheet. Teachers were also asked how they explained adding like terms to their students. Immediately after students completed the algebraic tasks, the second interviews with teachers were conducted. At that time, teachers actually created their library of student work and then added to, or revised, their initial WOT sheets. The teachers brought class sets of their students’ attempts at the algebraic tasks, homework assigned from the *Say It With Symbols* (Lappan et al., 1998) text, and any quizzes that they gave. Then, they selected a subset of these for their library of student samples to share with other teachers. About a month after the study, the teachers were given the opportunity to examine their library of student work again, and to add to or revise their WOT sheets one more time. The teachers were also given the opportunity to add to their library of student work, but none exercised that option. The problem posed at the final team meeting about how to explain adding like terms was also discussed. The teachers were asked to choose which method they considered best to explain adding like terms. All meetings with teachers and individual interviews were audio-taped and transcribed.

Data analysis

Data analysis drew upon a grounded theory approach (Strauss & Corbin, 1998) in which data are systematically gathered and analysed throughout the research process, and theory is derived from the data.

Analysis within this study was a continual process of simultaneously coding and analysing data and began during the initial interviews. Consistent with the models and modelling perspective described earlier, the records the teachers produced in the form of iterations of the WOT sheets and the library of student work helped produce a continuous trail of documentation. These artifacts were used to reflect on the nature of the teachers' developing models (Lesh & Kelly, 1999). The iterative cycles prompted the teachers and researcher to test, revise, and extend their knowledge development. From this process, a profile of each teacher was developed in which the information the teacher acquired and how they interpreted that information were described.

Results

Bruce's model of his students' algebraic thinking developed from a rather sparse model to include much richer interpretations, including well known results from research about students' algebraic thinking. This development is reflected in the three results of the study. First, Bruce developed a clearer understanding of the role of a variable in algebraic instruction. Second, he came to specifically embrace the use of visual representations in the context of teaching equivalent expressions. Third, Bruce demonstrated a high level of attention to details relative to other teachers in the study. These results are intimately connected with each other. Bruce initially supported the new curriculum. As the study progressed and his understanding of the role of a variable and algebraic representations grew, he continued to support the implementation of the curriculum. In turn, Bruce became increasingly aware of the details within students' solutions and the multiple ways his students solved the given tasks.

Before implementing the series of tasks, Bruce explained that he neither had learned algebra the way the curriculum advocated nor had he ever taught equivalent expressions within a context. In the beginning of the study, he described himself as "looking forward to it [teaching the unit] and at the same time I am a little apprehensive." When Bruce was prompted to start his first WOT sheet, he did not include any hints about the students' mathematical thinking, but he did offer the following two potential student mistakes:

- Students won't use inverse operations to correctly solve equations for variables.
- Students won't correctly define a variable and write a corresponding equation.

Both of these mistakes are related to solving equations, and not necessarily linked to the series of tasks on equivalent expressions. Solving equations does appear at the end of the *Say It With Symbols* unit (Lappan et al., 1998), but is not a part of these tasks. Bruce had a copy of the tasks in front of him when he started his first WOT sheet. Perhaps he was thinking globally, allowing him to be open to variations in student work. Alternatively, he may not have examined the material on equivalent expressions in detail. As demonstrated below, Bruce became increasingly focused on his students' thinking and, as his WOT sheets and library of student work progressed through the three cycles of iterative refinement, his responses became more specific to the algebraic material related to equivalent expressions.

The role of a variable

During the course of the study Bruce developed a clearer understanding of the role of a variable in the tasks in the instructional materials and beginning algebraic instruction. First, he recognised that the transition from arithmetic to algebra is complex, and that his students preferred numerical answers. Second, he recognised his students' tendencies to conjoin expressions. Third, he began to question a part of his past practice and the use of the pervasive "apples and oranges" metaphor in explaining how to add like terms.

Transition from arithmetic to algebra

When interviewed initially about the students' transition from arithmetic (numeric answers) to algebra, Bruce responded that his instructional approach was one of "constant practice with substituting numbers in, or with variables showing how the letters just represent numbers." Here, Bruce saw that the variables represented unknown quantities. He reported that he used "an arithmetic sentence, and then one that would correspond with it using variables to represent each of those numbers." In this instance, the number sentence $3 + 4 = 4 + 3$ would also be shown as $a + b = b + a$. Here, the symbols represented generalised numbers.

During the second interview, Bruce commented that the use of variables was difficult for his students to learn "because it is new to them. They haven't been dealing with variables for too long" and when his students did solve algebraic equations, "they [were] used to having numbers as answers." For example, his students were familiar with $2x + 4 = 10$ and the numerical answer of $x = 3$, but not an expression such as $4s + 4$ as an answer (with a variable in the answer). When asked again how he helped students make the transition, he responded: "We plug in numbers for *s* that would make s equal to some value." This response indicated that Bruce was not helping his students accept algebraic expressions for solutions because he was suggesting to his students that they find a specific numerical answer. Bruce had a sense that the transition from arithmetic to algebra took time, but perhaps not a clear sense of how to help his students accept algebraic expressions as solutions. During the implementation of the series of tasks, Bruce found that his students preferred a numerical answer. Bruce noted that his students could readily find the exact number of tiles surrounding a given sized pool, but that they had difficulty writing algebraic equations. After the tasks were completed, Bruce specifically added this hint to his second WOT sheet:

- It was difficult for students to find algebraic equations for the number of 1-foot square tiles surrounding a square pool.

Bruce recognised his students' desire for numerical answers, but he did not fully articulate that on his WOT sheet.

At the conclusion of this study, Bruce did acknowledge the role of algebraic expressions standing alone. Using a specific problem from the series of tasks, he was asked about his students' understanding of a variable in the expressions $4(s + 1)$, or $4s + 4$. He responded:

> …that "*s*" represented the side length of the pool and I think most of them [the students] did get that meaning. So when they see the variable in that context I think that they would know that that's what that meant. As far as putting a variable into an equation like $2x + 5 = 19$ or whatever, they know the x is something they are just going to have to solve for it.

While Bruce quickly made his own jump to solving for a variable, he distinguished between an algebraic expression as a solution and solving an equation for a numerical solution. Bruce further commented:

> The transition is I think something that has got to be taken, transition from arithmetic to algebra, something that has got to be taken I think a little more seriously. You know, more slowly and more seriously than what I have done in the past. I know that I will be doing this again. I will never do it the way I did on the past. I think this is better.

Bruce decided not to teach equivalent expressions with his former direct instruction approach but rather he would use this series of student tasks. Bruce was aware that his own ideas shifted during the series of lessons.

Students' tendencies to conjoin expressions

By the end of the study, Bruce was able to articulate his students' tendencies to conjoin expressions. Bruce added this hint to his last WOT sheet:

- Students might add $16x + 2$ together.

Bruce described that his students added together or conjoined expressions such as $16x + 2$ to equal $18x$ when they were solving equations. Bruce did not add this thought to his WOT sheets until the last iteration, about a month after the series of lessons was taught. By that time his students had moved into solving equations, but Bruce thought that it was now significant to add this hint about his students' algebraic thinking as a salient feature to be aware of for the entire series of tasks. Although he may have been aware of that tendency, Bruce did not express it as a significant element of beginning algebraic instruction until the third iteration of his WOT sheets.

Questioning the use of a metaphor

By the end of the study, Bruce also began to question his own use of the pervasive but confusing "apples and oranges" metaphor that many teachers use in explaining adding like terms. Bruce initially used this metaphor with his students as an explanation for adding like terms. He explained, "apples and oranges don't mix when you are adding them up, so x's and y's don't mix… because they are not the same thing." When initially prompted about a context, Bruce answered he had never thought of explaining how to add like terms in a context. During the last team discussion, three methods of explaining how to add like terms were discussed (see data collection). During the final individual interview, when Bruce was presented with the same three methods, he decided that the bank account problem, or the area and line problem, would be a better way to explain adding like terms. Bruce noted that the apples and oranges way assumed that sometimes the variables represent objects or nouns, but the other methods for explaining moved "away from this noun thing." He further commented:

> When I started teaching stuff like this combining like terms, I would just never have thought of something like that in a million years, but that's a great idea… but nothing works for all kids… you have to present things in various ways because everybody is different.

Bruce's understanding of the role of a variable shifted during the course of the study. In the beginning, Bruce related to variables in an equation as unknowns to be solved. By the end of the study, Bruce articulated his awareness of expressions as solutions, and connected the variables in given expressions to spatial contexts. He was aware of students' tendencies to conjoin expressions. Bruce also thought more deeply about the transition from arithmetic to algebra and questioned his own use of the "apples and oranges" metaphor.

Visual representations

During the study, Bruce, for the first time, employed visual representations of equivalent expressions. In the past he had taught equivalent expressions in a procedural manner. While he taught the series of tasks, Bruce began to perceive the usefulness of visual representations in demonstrating the equivalence of two expressions. This is demonstrated through selections in Bruce's library of student work, excerpts from his interviews, and by the additions to his WOT sheets.

During the implementation phase, Bruce selected student work samples to support his observations. Bruce considered the following selection to be an exemplary piece of work connecting the algebraic expressions to the drawings. The selection represents a student's solution to the tiling pools problem (as described previously). For Bruce, the student's solution helped reinforce the power of representing the solution visually. Student A drew squares representing corners, and she put a "1" in those sections to clearly demarcate the corners (see Figure 3). The student did not answer part e. of the task, but Bruce focused on the quality of the visual aspects of the problem.

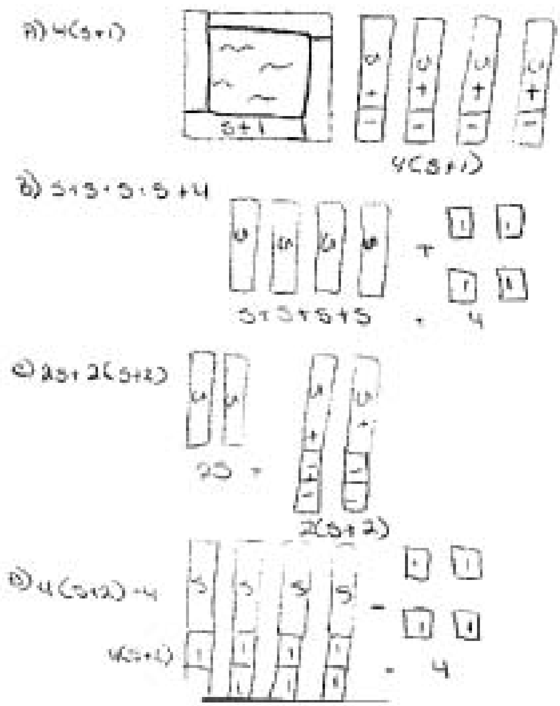


Figure 3. Bruce's library of student work: Student A

Bruce also observed that some students still operated at a procedural level. Student M began with a reasonable representation in part a. of the task, but drew representations without the corners in part b (see Figure 4). Bruce observed that this is not necessarily incorrect. However, in parts c. and d., Student M used a rectangle to represent the side in the parenthesis, but did not accurately represent the given expression. The student simply rewrote the expression substituting a small rectangle for the "*s*." This student's explanation in part e. read, "They are all equivalent to Takashi's [a given example in the text] expression because we started with the same amount of border tiles as he did and ended with the same amount." Bruce did not comment on the student's explanation, but the sentence does not fully describe the equivalence, or use grade level appropriate mathematical vocabulary using algebraic reasoning, tables, graphs, or symbol manipulation as suggested by the text. Bruce thought that the mistakes in Figure 4 pointed towards a developing yet incomplete understanding of the role of the variable "*s*".

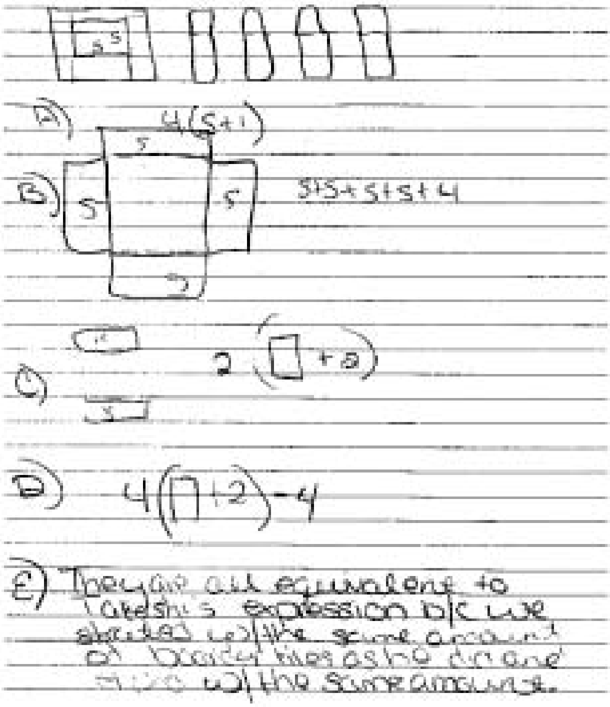


Figure 4. Bruce's library of student work: Student M

Bruce found that many, but not all, of his students answered the given problem correctly, and their visual representations reflected a clear understanding of the side and the corners in the problems. Bruce's ideas about visual representations were further supported by his thoughts during the interviews. He enthusiastically described an episode in his classroom where students connected the visual representations and the use of variables:

> For example, having a student come to the board to explain how they are dividing up the border of tiles around that pool … breaking up that border into pieces…[And] relating the pieces to the parts of the expression that we used to calculate how many tiles go around the border of the pool.

Bruce observed that his students connected the visual representation with the relevant parts of a given algebraic expression. Bruce also noted that:

> I did have him show me his [a student's] expression and he did for the s + 2… on both ends with two tiles and then again down here, the two sides… I was surprised [because of] the student that it was. He is one of my students who is generally not that "with it".

Bruce's efforts to teach the series of tasks were reinforced when he saw that more of his students than usual embodied the algebraic ideas in the visual representations.

Immediately after teaching the series of tasks, Bruce added the following thoughts about the students' mathematical thinking to his WOT sheet:

- Using a visual or geometric representation helps the students.
- Students incorrectly labelled the corners in their representations of the different expressions for the number of tiles.

His model now included the context of the visual representation of the problems. In his last iteration of his WOT sheets, Bruce again stated emphatically that using a visual or geometric representation helped the students understand the equivalence of two expressions. At this juncture, Bruce commented, "They are using pictures and diagrams and they are labelling the different parts of the diagrams with variable terms, and then expressing those areas in different ways." Overall, Bruce perceived that his students could now use visual representations to show that two expressions were equivalent and this helped his students make sense of the algebraic expressions. Bruce clearly recognised that the use of visual representations became a powerful tool for his students.

Attention to detail

Bruce was beginning to develop significant insight into his students' algebraic thinking as shown by his responses in his library of student work, interviews, and WOT sheets. Also, Bruce's attention to details in his library of student work continued to grow. Bruce administered a quiz, after teaching the tiling pools problem, where students could formulate their answers using any strategy that made sense to them. Four different student strategies are included here as representative of the details that Bruce desired in his library of student work. His recognition of the potential variety of solutions stands in contrast to his limited thoughts on his first WOT sheet.

First, Bruce noted that Student P appeared to count the number of tiles because small dots from a pencil tip appeared in most of the squares in the diagram (see Figure 5). These dots did not copy well but appeared in the original paper. This student also used multiplication (in the upper left corner) and then added on the four corners. Although Bruce observed that the student had multiple ways to approach the problem, he did not comment that perhaps one method was used to do the problem, and another used to check.

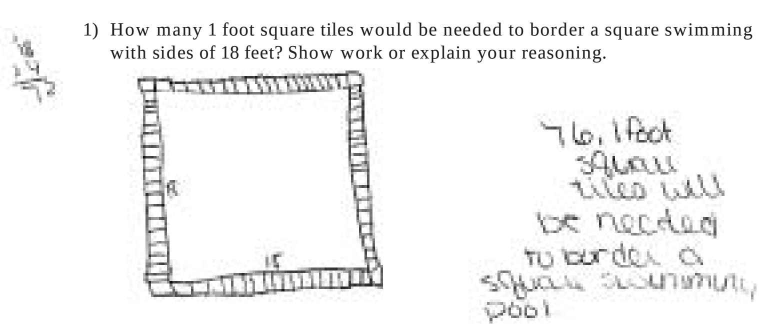


Figure 5. Bruce's Library of Student Work—Student P

In a second student sample (Figure 6), Student N solved the problem by applying an algebraic expression and included a diagram. Bruce thought that this work represented a good solution. However, Bruce did not note that the student labelled each corner in a manner that might signify confusion between linear units and area. For example, Student N placed a "1" in the corner which might represent a length of 1 for the side of one tile or, alternatively, the area of a 1 x 1 tile. Likewise, the way Student N placed the "18" might represent a length of 18 individual square tiles, or possibly an area of 18.

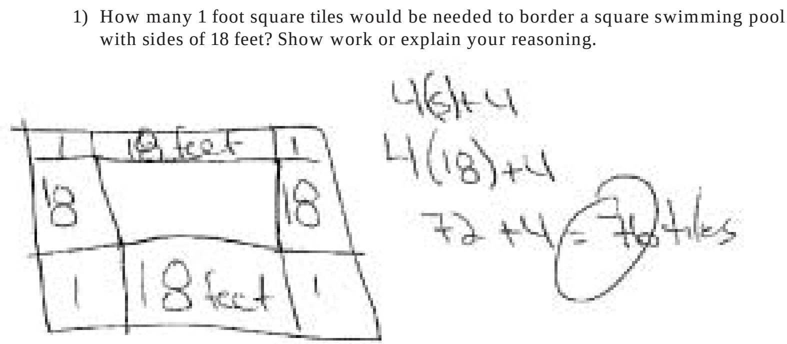


Figure 6. Bruce's Library of Student Work—Student N

Third, Bruce noted that Student J used an algebraic expression to solve the problem and wrote a thorough explanation which is included in Figure 7.

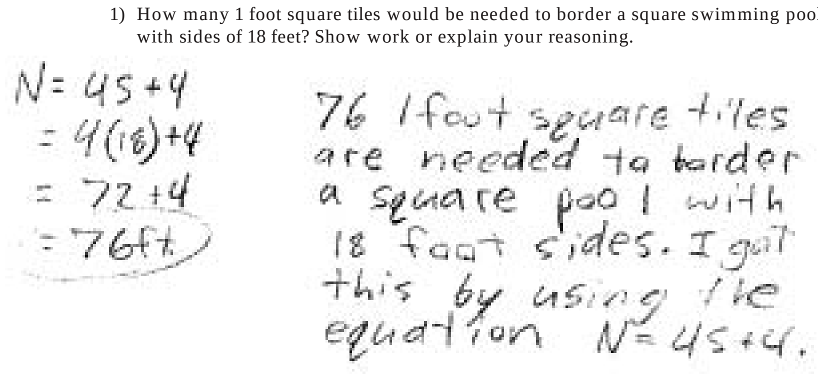


Figure 7. Bruce's Library of Student Work—Student J

Last, student O used a table of values (see Figure 8). Bruce commented that he would allow any mathematically correct solution, and that for this grade level this type of solution was acceptable.

Bruce was impressed by the differences in the students' solutions and specifically wanted this variety of responses to be a part of his library of student work. Bruce became more aware of the multiple ways students can solve these types of problems. Bruce acquired a significant amount of information about his students' thinking through examining their work as previously described in this paper. However in several instances (see Figures 4, 6, & 7) he may not have acquired all the available information.

1) How many 1 foot square tiles would be needed to border a square swimming pool with sides of 18 feet? Show work or explain your reasoning.

Figure 8. Bruce's Library of Student Work–Student O

Discussion and concluding points

This study illuminates how one teacher, Bruce, interpreted his own practice when teaching equivalent expressions. Through Bruce's participation in two model-eliciting activities, his model of his students' algebraic thinking developed in ways that demonstrated his understanding of the diversity in their thinking about equivalent expressions.

Four aspects related to pedagogical content knowledge (Shulman, 1986) surfaced during the study. First, Bruce started to become aware that the transition from arithmetic to algebra takes time. It is known that the transition from thinking procedurally to thinking conceptually takes time and that teachers often operate on a conceptual level when students lag behind at a procedural level (Kieran, 1992). Bruce questioned his own teaching during this transition and decided that he "would never teach it that [the procedural] way again." Second, Bruce started to articulate his awareness of well known student errors, including his students' desire for numerical answers, and for conjoining expressions. Third, Bruce came to appreciate the benefits of connecting geometric representations to the teaching of equivalent expressions. Bruce was initially “apprehensive” about such an approach, but he perceived the usefulness of the visual representations as the study progressed. Bruce began to allow the use of the visual representation to help drive the students’ understandings of the algorithm. A fourth issue related to pedagogical content knowledge concerned the robust nature of the “apples and oranges” or “fruit salad” metaphor and the well-known results about students’ confusion in understanding the role of a variable. The curriculum materials used in this study never referenced this metaphor. Bruce began to question the use of the “fruit salad” approach in his instruction through participation in the study. The model-eliciting activities and related conversations with the researcher and his colleagues brought Bruce to reconsider the use of this long standing approach to adding like terms.

The findings of this study suggest two implications for instruction. First, the students’ tasks and the teachers’ model-eliciting activities guided Bruce and his students to using variables in contexts different from those that had been used in the past. The rich nature of the series of tasks from the reform-based curriculum differed from those found in most traditional middle school classrooms and curricula. Rich student tasks coupled with teachers’ participation in model-eliciting activities have the potential to enhance teachers’ models of their students’ algebraic thinking. Second, consistent with prior research on teachers’ examinations of students’ work, Bruce attended to many details, but perhaps not all the details of his students’ representations. The existence of additional information suggests that he (and perhaps other teachers) might benefit from additional experiences in creating libraries of student work. It is not just the initial selection process of a library of student work that is important. Analysing, revising, and refining the library are necessary components to increase the capacity of teachers to acquire information about their students’ algebraic thinking as demonstrated by the development of Bruce’s model.

Bruce was able to articulate an increasingly expanded model of his students’ algebraic thinking about equivalent expressions. The findings of the study and its related professional development model have the potential to improve in-service instruction by guiding teachers into more useful ways of thinking about algebraic instruction. Further research is warranted on teachers’ participation in these types of professional development activities to determine the scope of its effectiveness.

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